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Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals

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Abstract

Some Hermite-Hadamard type inequalities for generalized k-fractional integrals (which are also named (k,s)-Riemann-Liouville fractional integrals) are obtained for a fractional integral, and an important identity is established. Also, by using the obtained identity, we get a Hermite-Hadamard type inequality.

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1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as a Hermite-Hadamard integral inequality for convex functions [1].

Sarikaya et al. established the following results for Riemann-Liouville fractional integrals.

Theorem 1.1 (see Theorem 2 in [2]) *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

with $\alpha > 0$, where the symbols $J_{a^+}^\alpha$ and $J_{b^-}^\alpha$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in \mathbb{R}^+$ that are defined by

$$J_{a^+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}\,dt \quad (0 \leq a < x \leq b)$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(t)(t-x)^{\alpha-1}\,dt \quad (0 \le a \le x < b)$$

respectively. Here $\Gamma(\cdot)$ *denotes the classical gamma function* [3], *Chapter* 6.

Theorem 1.2 (see Theorem 3 in [2]) *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on* (a,b) *with* $a < b$. *If* $f' \in L[a,b]$, *then the following inequality for Riemann-Liouville fractional integrals holds*:

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right]\right| \\ &\quad \le \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right) \end{aligned} \tag{1.3}$$

with $\alpha > 0$.

The Pochhammer k-symbol $(x)_{n,k}$ and the k-gamma function Γ_k are defined as follows (see [4]):

$$(x)_{n,k} := x(x+k)(x+2k)\cdots\big(x+(n-1)k\big) \quad (n \in \mathbb{N}; k > 0) \tag{1.4}$$

and

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \quad \big(k > 0; x \in \mathbb{C} \setminus k\mathbb{Z}_0^-\big), \tag{1.5}$$

where $k\mathbb{Z}_0^- := \{kn : n \in \mathbb{Z}_0^-\}$. It is noted that the case $k = 1$ of (1.4) and (1.5) reduces to the familiar Pochhammer symbol $(x)_n$ and the gamma function Γ. The function Γ_k is given by the following integral:

$$\Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}}\,dt \quad \big(\Re(x) > 0\big). \tag{1.6}$$

The function Γ_k defined on $\mathbb{R}^+$ is characterized by the following three properties: (i) $\Gamma_k(x+k) = x\Gamma_k(x)$; (ii) $\Gamma_k(k) = 1$; (iii) $\Gamma_k(x)$ is logarithmically convex. It is easy to see that

$$\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \quad \big(\Re(x) > 0; k > 0\big). \tag{1.7}$$

We want to recall the preliminaries and notations of some well-known fractional integral operators that will be used to obtain some remarks and corollaries.

The (k,s)-Riemann-Liouville fractional integral operator ${}_k^s\mathcal{J}_a^{\alpha}$ of order $\alpha > 0$ for a real-valued continuous function $f(t)$ is defined as (see [5], p.79, 2.1. Definition):

$${}_k^s\mathcal{J}_a^{\alpha} f(x) = \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x \big(x^{s+1} - t^{s+1}\big)^{\frac{\alpha}{k}-1} t^s f(t)\,dt, \tag{1.8}$$

where $k > 0$, $\beta > 0$ and $s \in \mathbb{R} \setminus \{-1\}$.

The most important feature of (k,s)-fractional integrals is that they generalize some types of fractional integrals (Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, generalized fractional integral and Hadamard fractional integral). These important special cases of the integral operator ${}^{s}_{k}\mathcal{J}^{\alpha}_{a}$ are mentioned below.

(1) For $k = 1$, the operator in (1.8) yields the following generalized fractional integrals defined by Katugompola in [6]:

$$ {}^{r}_{a}\mathcal{J}^{\alpha}_{t} f(x) = \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \left(x^{r+1} - t^{r+1}\right)^{\alpha-1} t^r f(t)\,dt. \tag{1.9} $$

(2) Firstly by taking $k = 1$, after that by taking limit $r \to -1^+$ and using L'Hôpital's rule, the operator in (1.8) leads to the Hadamard fractional integral operator [1, 7]. That is,

$$ \begin{aligned} \lim_{r\to -1^+} {}^{r}_{a}\mathcal{J}^{\alpha}_{t} f(x) &= \lim_{r\to -1^+} \frac{(r+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)t^r}{(x^{r+1}-t^{r+1})^{1-\alpha}}\,dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x \lim_{r\to -1^+} f(t)t^r \left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha} dt \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \lim_{r\to -1^+} \left(\frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha} \frac{dt}{t} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \left(\lim_{r\to -1^+} \frac{r+1}{x^{r+1}-t^{r+1}}\right)^{1-\alpha} \frac{dt}{t} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log\frac{x}{t}\right) f(t) \frac{dt}{t} \\ &= {}_{H}\mathcal{J}^{\alpha}\big[f(t)\big] \end{aligned} \tag{1.10} $$

(see [8], p.569, eq. (3.13)).

(3) If we take $s = 0$ in (1.8), operator (1.8), reduces to the k-Riemann-Liouville fractional integral operator, which has been firstly defined by Mubeen and Habibullah in [9]. This relation is as follows:

$$ \mathcal{J}^{\alpha}_{a,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)\,dt. \tag{1.11} $$

(4) Again, taking $s = 0$ and $k = 1$, operator (1.8) gives us the Riemann-Liouville fractional integration operator

$$ J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt. \tag{1.12} $$

In recent years, these fractional operators have been studied and used to extend especially Grüss, Chebychev-Grüss and Pólya-Szegö type inequalities. For more details, one may refer to the recent works and books [7, 10–21].

2 Main results

Let $f : I^\circ \to \mathbb{R}$ be a given function, where $a, b \in I^\circ$ and $0 < a < b < \infty$. We suppose that $f \in L_\infty(a,b)$ such that ${}^{s}_{k}J^{\alpha}_{a^+} f(x)$ and ${}^{s}_{k}J^{\alpha}_{b^-} f(x)$ are well defined. We define functions

$$ \tilde{f}(x) := f(a+b-x), \quad x \in [a,b] $$

and

$$F(x) := f(x) + \tilde{f}(x), \quad x \in [a, b].$$

Hermite-Hadamard's inequality for convex functions can be represented in a (k, s)-fractional integral form as follows by using the change of variables $u = \frac{t-a}{x-a}$; we have from (1.8)

$$\begin{aligned} {}_k^s\mathcal{J}_a^\alpha f(x) = (x-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ux+(1-u)a)^s}{((ux+(1-u)a)^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1}} \\ \times f\bigl(ux + (1-u)a\bigr)\, ds, \end{aligned} \tag{2.1}$$

where $x > a$.

Theorem 2.1 *Let $\alpha, k > 0$ and $s \in \mathbb{R} \setminus \{-1\}$. If f is a convex function on $[a, b]$, then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1} - a^{s+1})^{\frac{\alpha}{k}}}\bigl[{}_k^sJ_{a^+}^\alpha F(b) + {}_k^sJ_{b^-}^\alpha F(a)\bigr] \\ &\leq \frac{f(a) + f(b)}{2}. \end{aligned} \tag{2.2}$$

Proof For $u \in [0, 1]$, let $\xi = au + (1-u)b$ and $\eta = (1-u)a + bu$. Using the convexity of f, we get

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{\xi+\eta}{2}\right) \leq \frac{1}{2}f(\xi) + \frac{1}{2}f(\eta).$$

That is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\bigl(au + (1-u)b\bigr) + \frac{1}{2}f\bigl((1-u)a + bu\bigr). \tag{2.3}$$

Now, multiplying both sides of (2.3) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \frac{(ub+(1-u)a)^s}{[b^{s+1} - (ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0, 1)$ with respect to u, we get

$$\begin{aligned} &(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{(ub+(1-u)a)^s\, du}{[b^{s+1} - (ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}} \\ &\quad\leq \frac{1}{2}(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ub+(1-u)a)^s f(au+(1-u)b)\, du}{[b^{s+1} - (ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}} \\ &\qquad+ \frac{1}{2}(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(ub+(1-u)a)^s f((1-u)a+bu)\, du}{[b^{s+1} - (ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}. \end{aligned}$$

Note that we have

$$\int_0^1 \frac{(ub+(1-u)a)^s\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}} = \frac{k(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{\alpha(s+1)(b-a)}.$$

Using the identity

$$\tilde{f}\big((1-u)a+bu\big) = f\big(au+(1-u)b\big),$$

and from (2.1), we obtain

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(ub+(1-u)a)^s f(au+(1-u)b)\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}} = {}^s_kJ^\alpha_{a^+}\tilde{f}(b)$$

and

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(ub+(1-u)a)^s f((1-u)a+bu)\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}} = {}^s_kJ^\alpha_{a^+}f(b).$$

Accordingly, we have

$$\frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} f\left(\frac{a+b}{2}\right) \le \frac{{}^s_kJ^\alpha_{a^+}F(b)}{2}. \tag{2.4}$$

Similarly, multiplying both sides of (2.3) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(ub+(1-u)a)^s}{[(bu+(1-u)a)^{s+1}-a^{s+1}]^{1-\frac{\alpha}{k}}},$$

integrating over $(0,1)$ with respect to u, and from (2.1), we also get

$$\frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} f\left(\frac{a+b}{2}\right) \le \frac{{}^s_kJ^\alpha_{b^-}F(a)}{2}. \tag{2.5}$$

By adding inequalities (2.4) and (2.5), we get

$$f\left(\frac{a+b}{2}\right) \le \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\big[{}^s_kJ^\alpha_{a^+}F(b) + {}^s_kJ^\alpha_{b^-}F(a)\big],$$

which is the left-hand side of inequality (2.2).

Since f is convex, for $u \in [0,1]$, we have

$$f\big(au+(1-u)b\big) + f\big((1-u)a+bu\big) \le f(a)+f(b). \tag{2.6}$$

Multiplying both sides of (2.6) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(ub+(1-u)a)^s}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to u, we get

$$\begin{aligned}
&(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(ub+(1-u)a)^s f(au+(1-u)b)\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}\\
&\quad+(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^1 \frac{(ub+(1-u)a)^s f((1-u)a+bu)\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}\\
&\quad\le (b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\big[f(a)+f(b)\big]\int_0^1 \frac{(ub+(1-u)a)^s\,du}{[b^{s+1}-(ub+(1-u)a)^{s+1}]^{1-\frac{\alpha}{k}}}.
\end{aligned}$$

That is,

$$ {}_k^sJ_{a^+}^{\alpha}F(b)\le \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\big[f(a)+f(b)\big]. \tag{2.7}$$

Similarly, multiplying both sides of (2.6) by

$$(b-a)\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\frac{(ub+(1-u)a)^s}{[(ub+(1-u)a)^{s+1}-a^{s+1}]^{1-\frac{\alpha}{k}}}$$

and integrating over $(0,1)$ with respect to u, we also get

$$ {}_k^sJ_{b^-}^{\alpha}F(a)\le \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}\big[f(a)+f(b)\big]. \tag{2.8}$$

Adding inequalities (2.7) and (2.8), we obtain

$$\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\big[{}_k^sJ_{a^+}^{\alpha}F(b)+{}_k^sJ_{b^-}^{\alpha}F(a)\big]\le\frac{f(a)+f(b)}{2},$$

which is the right-hand side of inequality (2.2). So the proof is complete. □

We want to give the following function that we will use later: For $\alpha,k>0$ and $s\in\mathbb{R}\setminus\{-1\}$, let $\nabla_{\alpha,s}:[0,1]\to\mathbb{R}$ be the function defined by

$$\begin{aligned}
\nabla_{\alpha,s}(t):={}&\big(\big(ta+(1-t)b\big)^{s+1}-a^{s+1}\big)^{\frac{\alpha}{k}}-\big(\big(bt+(1-t)a\big)^{s+1}-a^{s+1}\big)^{\frac{\alpha}{k}}\\
&+\big(b^{s+1}-\big(tb+(1-t)a\big)^{s+1}\big)^{\frac{\alpha}{k}}-\big(b^{s+1}-\big(ta+(1-t)b\big)^{s+1}\big)^{\frac{\alpha}{k}}.
\end{aligned}$$

In order to prove our main result, we need the following identity.

Lemma 2.1 *Let $\alpha,k>0$ and $s\in\mathbb{R}I^{\circ}$. If f is a differentiable function on I° such that $f'\in L[a,b]$ with $a<b$, then we have the following identity*:

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\big[{}_k^sJ_{a^+}^{\alpha}F(b)+{}_k^sJ_{b^-}^{\alpha}F(a)\big]\\
&\quad=\frac{(b-a)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\int_0^1\nabla_{\alpha,s}(t)f'\big(ta+(1-t)b\big)\,dt.
\end{aligned}\tag{2.9}$$

Proof Using integration by parts, we obtain

$$
\begin{aligned}
{}_k^sJ_{a^+}^{\alpha}F(b) = {}& \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}F(a) + \frac{(b-a)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \\
& \times \int_0^1 \bigl[\bigl(b^{s+1}-\bigl(bu+(1-u)a\bigr)^{s+1}\bigr)\bigr]^{\frac{\alpha}{k}} F'\bigl(bu+(1-u)a\bigr)\,du.
\end{aligned}
\tag{2.10}
$$

Similarly, we get

$$
\begin{aligned}
{}_k^sJ_{b^-}^{\alpha}F(a) = {}& \frac{(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}F(b) - \frac{(b-a)}{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)} \\
& \times \int_0^1 \bigl[\bigl(bu+(1-u)a\bigr)^{s+1}-a^{s+1}\bigr]^{\frac{\alpha}{k}} F'\bigl(bu+(1-u)a\bigr)\,du.
\end{aligned}
\tag{2.11}
$$

Using the fact that $F(x) = f(x) + \tilde{f}(x)$ and by simple computation, from equalities (2.10) and (2.11), we get

$$
\begin{aligned}
& \frac{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}{(b-a)}\left(\frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\bigl[{}_k^sJ_{a^+}^{\alpha}F(b) + {}_k^sJ_{b^-}^{\alpha}F(a)\bigr]\right) \\
& \quad = \int_0^1 \bigl[\bigl(\bigl(bu+(1-u)a\bigr)^{s+1}-a^{s+1}\bigr)^{\frac{\alpha}{k}} - \bigl(b^{s+1}-\bigl(bu+(1-u)a\bigr)^{s+1}\bigr)^{\frac{\alpha}{k}}\bigr] \\
& \qquad \times F'\bigl(bu+(1-u)a\bigr)\,du.
\end{aligned}
\tag{2.12}
$$

Note that we have

$$
F'\bigl(bu+(1-u)a\bigr) = f'\bigl(bu+(1-u)a\bigr) - f'\bigl(au+(1-u)b\bigr), \quad u\in[0,1].
$$

Then we can easily obtain

$$
\begin{aligned}
& \int_0^1 \bigl(\bigl(bu+(1-u)a\bigr)^{s+1}-a^{s+1}\bigr)^{\frac{\alpha}{k}} F'\bigl(bu+(1-u)a\bigr)\,du \\
& \quad = \int_0^1 \bigl(\bigl(ta+(1-t)b\bigr)^{s+1}-a^{s+1}\bigr)^{\frac{\alpha}{k}} f'\bigl(ta+(1-t)b\bigr)\,dt \\
& \qquad - \int_0^1 \bigl(\bigl(bt+(1-t)a\bigr)^{s+1}-a^{s+1}\bigr)^{\frac{\alpha}{k}} f'\bigl(ta+(1-t)b\bigr)\,dt
\end{aligned}
\tag{2.13}
$$

and

$$
\begin{aligned}
& \int_0^1 \bigl(b^{s+1}-\bigl(bu+(1-u)a\bigr)^{s+1}\bigr)^{\frac{\alpha}{k}} F'\bigl(bu+(1-u)a\bigr)\,du \\
& \quad = \int_0^1 \bigl(b^{s+1}-\bigl(ta+(1-t)b\bigr)^{s+1}\bigr)^{\frac{\alpha}{k}} f'\bigl(ta+(1-t)b\bigr)\,dt \\
& \qquad - \int_0^1 \bigl(b^{s+1}-\bigl(bt+(1-t)a\bigr)^{s+1}\bigr)^{\frac{\alpha}{k}} f'\bigl(ta+(1-t)b\bigr)\,dt.
\end{aligned}
\tag{2.14}
$$

Thus, the desired inequality (2.9) follows from inequalities (2.12), (2.13) and (2.14). □

For $\alpha, k > 0$, we introduce the following operator:

$$\Im(s,x,y) := \int_a^{\frac{a+b}{2}} |x-u|\left|y^{s+1}-u^{s+1}\right|^{\frac{\alpha}{k}}\,du - \int_{\frac{a+b}{2}}^b |x-u|\left|y^{s+1}-u^{s+1}\right|^{\frac{\alpha}{k}}\,du,$$

$s \in \mathbb{R} \setminus \{-1\}$, $x, y \in [a,b]$.

Using Lemma 2.1, we can obtain the following (k,s)-fractional integral inequality.

Theorem 2.2 *Let $\alpha, k > 0$ and $s \in \mathbb{R} \setminus \{-1\}$. If f is a differentiable function on I° such that $f' \in L[a,b]$ with $a < b$ and $|f'|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}^s_kJ^\alpha_{a^+}F(b) + {}^s_kJ^\alpha_{b^-}F(a)\right]\right| \\ \leq \frac{\Psi(s,\alpha,a,b)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}(b-a)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right), \tag{2.15}$$

where

$$\Psi(s,\alpha,a,b) = \Im(s,b,b) + \Im(s,a,b) - \Im(s,b,a) - \Im(s,a,a).$$

Proof Using Lemma 2.1 and the convexity of $|f'|$, we obtain

$$\left|\frac{f(a)+f(b)}{2} - \frac{(s+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left[{}^s_kJ^\alpha_{a^+}F(b) + {}^s_kJ^\alpha_{b^-}F(a)\right]\right| \\ \leq \frac{(b-a)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\int_0^1 \left|\nabla_{\alpha,s}(t)\right|\left|f'\big(ta+(1-t)b\big)\right|\,dt \\ \leq \frac{(b-a)}{4(b^{s+1}-a^{s+1})^{\frac{\alpha}{k}}}\left(\left|f'(a)\right|\int_0^1 t\left|\nabla_{\alpha,s}(t)\right|\,dt + \left|f'(b)\right|\int_0^1 (1-t)\left|\nabla_{\alpha,s}(t)\right|\,dt\right). \tag{2.16}$$

Note that

$$\int_0^1 t\left|\nabla_{\alpha,s}(t)\right|\,dt = \frac{1}{(b-a)^2}\int_a^b \left|\wp(u)\right|(b-u)\,du,$$

where

$$\wp(u) = \left(u^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}} - \left((b+a-u)^{s+1}-a^{s+1}\right)^{\frac{\alpha}{k}} \\ + \left(b^{s+1}-(b+a-u)^{s+1}\right)^{\frac{\alpha}{k}} - \left(b^{s+1}-u^{s+1}\right)^{\frac{\alpha}{k}}, \quad u \in [a,b].$$

Observe that $\wp$ is a non-decreasing function on $[a,b]$. Moreover, we have $\wp(a) = -2(b^{s+1} - a^{s+1})^{\frac{\alpha}{k}} < 0$ and $\wp(\frac{a+b}{2}) = 0$. Thus, we have

$$\begin{cases} \wp(u) \leq 0 & \text{if } a \leq u \leq \frac{a+b}{2}, \\ \wp(u) > 0 & \text{if } \frac{a+b}{2} < u \leq b. \end{cases}$$

So, we obtain

$$(b-a)^2 \int_0^1 t\left|\nabla_{\alpha,s}(t)\right| dt = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4,$$

where

$$\zeta_1 = \int_a^{\frac{a+b}{2}} (b-u)\left(b^{s+1} - u^{s+1}\right)^{\frac{\alpha}{k}} du - \int_{\frac{a+b}{2}}^b (b-u)\left(b^{s+1} - u^{s+1}\right)^{\frac{\alpha}{k}} du,$$

$$\zeta_2 = -\int_a^{\frac{a+b}{2}} (b-u)\left(u^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} du + \int_{\frac{a+b}{2}}^b (b-u)\left(u^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} du,$$

$$\zeta_3 = \int_a^{\frac{a+b}{2}} (b-u)\left((b+a-u)^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} du - \int_{\frac{a+b}{2}}^b (b-u)\left((b+a-u)^{s+1} - a^{s+1}\right)^{\frac{\alpha}{k}} du,$$

$$\zeta_4 = -\int_a^{\frac{a+b}{2}} (b-u)\left(b^{s+1} - (b+a-u)^{s+1}\right)^{\frac{\alpha}{k}} du + \int_{\frac{a+b}{2}}^b (b-u)\left(b^{s+1} - (b+a-u)^{s+1}\right)^{\frac{\alpha}{k}} du.$$

Observe that $\zeta_1 = \Im(s,b,b)$ and $\zeta_2 = -\Im(s,b,a)$. Using the change of variable $v = a+b-u$, we get $\zeta_3 = -\Im(s,a,a)$ and $\zeta_4 = \Im(s,a,b)$. Thus, we obtain

$$\int_0^1 t\left|\nabla_{\alpha,s}(t)\right| dt = \frac{\Im(s,b,b) + \Im(s,a,b) - \Im(s,b,a) - \Im(s,a,a)}{(b-a)^2}. \tag{2.17}$$

Similarly,

$$\int_0^1 (1-t)\left|\nabla_{\alpha,s}(t)\right| dt = \frac{\Im(s,b,b) + \Im(s,a,b) - \Im(s,b,a) - \Im(s,a,a)}{(b-a)^2}. \tag{2.18}$$

So, the desired inequality (2.15) follows from inequalities (2.16), (2.17) and (2.18). □

3 Conclusions

Lastly, we conclude this paper by remarking that we have obtained a Hermite-Hadamard inequality, an identity and a Hermite-Hadamard type inequality for a generalized k-fractional integral operator. Therefore, by suitably choosing the parameters, one can further easily obtain additional integral inequalities involving the various types of fractional integral operators from our main results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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